

One loop boundary effects: techniques and applications. *

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Abstract

A pedagogical introduction to the heat kernel technique, zeta function and Casimir effect is presented. Several applications are considered. First we derive the high temperature asymptotics of the free energy for boson fields in terms of the heat kernel expansion and zeta function. Another application is chiral anomaly for local (MIT bag) boundary conditions. Then we rederive the Casimir energies for perfectly conducting rectangular cavities using a new technique. The new results for the attractive Casimir force acting on each of the two perfectly conducting plates inside an infinite perfectly conducting waveguide of the same cross section as the plates are presented at zero and finite temperatures.

1 Introduction

The main problem of the quantum field theory with boundaries is its renormalization and physical meaning of the results obtained. Divergences that appear in quantum field theory make the problems on manifolds with boundaries more complicated than in infinite space.

In the presence of boundaries or singularities the heat kernel technique is an effective tool for the analysis of the one loop effects (see reviews [1],

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[2]). Different applications of the heat kernel expansion exist. The heat kernel technique seems to be the easiest way for the calculation of quantum anomalies, calculation of effective actions based on finite-mode regularization and analysis of divergences in quantum field theory.

Chiral anomaly, which was discovered more than 35 years ago [3], still plays an important role in physics. On smooth manifolds without boundaries many successful approaches to the anomalies exist [4],[5], [6]. The heat kernel approach to the anomalies is essentially equivalent to the Fujikawa approach [7] and to the calculations based on the finite-mode regularization [8], but it can be more easily extended to complicated geometries. The local chiral anomaly in the case of non-trivial boundary conditions (MIT bag boundary conditions) has been calculated only recently [9].

Casimir effect [10] is a macroscopic quantum effect. Briefly speaking, if we impose classical boundary conditions on a quantum field on some boundary surface than we get the Casimir effect. There are several different physical situations that should be distinguished in the Casimir effect.

Suppose there are two spatially separated dielectrics, then in a dilute limit ($\epsilon \rightarrow 1$) the Casimir energy of this system is equal to the energy of pairwise interactions between dipoles of these two dielectrics via a Casimir-Polder retarded potential [11]. For a general case of separated dielectrics the Casimir energy can be calculated as in [12] or [13] (for a recent discussion of these issues see [14] and a review [15], new possible experiments in [16]).

A different situation takes place when there is a dilute dielectric ball or any other simply connected dielectric under study (see a review [17] for a discussion of related subjects and methods used). As it was pointed out in [18] and then discussed in detail in [19], microscopic interatomic distances should be taken into account to calculate the Casimir energy of a dilute dielectric ball. The average interatomic distance λ serves as an effective physical cut off for simply connected dielectrics.

The limit of a perfect conductivity ($\epsilon \rightarrow +\infty$) is opposite to a dilute case. This is the strong coupling limit of the theory. Any results obtained in this limit are nonperturbative ones.

The Casimir energy for a perfectly conducting rectangular cavity was first calculated in [20] using exponential regularization. Later it was derived by some other methods (see references and numerical analysis in [21], also a review [22]). In the present paper we derive the Casimir energy for rectangular cavities at zero temperature by a new method described in Sec.3. By use of this method we could rewrite the Casimir energy for rectangular cavities in the form that makes transparent its geometric interpretation. Also this method yields new exact results for the Casimir force acting on two or more perfectly conducting plates of an arbitrary cross section inside an infinite

perfectly conducting waveguide of the same cross section.

The paper is organized as follows. In Sec.2 we give an introduction to the formalism of the heat kernel and heat kernel expansion. Also we introduce a ζ -function [23] and calculate the one loop effective action in terms of ζ -function. Then we consider two examples. First we derive the high temperature expansion of the free energy for boson fields [24] in terms of the heat kernel expansion and ζ -function. Then we derive a chiral anomaly in four dimensions for an euclidean version of the MIT bag boundary conditions [25]. Sec.3 is devoted to the Casimir effect for perfectly conducting cavities. In Sec.3.1 we introduce a regularization and a convenient new method of calculations using an example of two perfectly conducting parallel plates, then apply it to more complicated rectangular geometries. We present the Casimir energy of the cavity in the form (91). Then we discuss an argument principle and ζ -functional regularization for the cavity. In Sec.3.6 we rewrite the Casimir energy of the cavity in terms of geometric optics (109). In Sec.3.7 we describe a possible experiment and derive the formula for the attractive force acting on each of the two parallel plates inside an infinite rectangular waveguide with the same cross section. Also we present a generalization of this result for the case of the two parallel perfectly conducting plates of an arbitrary cross section inside an infinite perfectly conducting waveguide with the same cross section at zero and finite temperatures.

2 Spectral techniques

2.1 Heat kernel

Consider a second order elliptic partial differential operator L of Laplace type on an n -dimensional Riemannian manifold. Any operator of this type can be expanded locally as

$$L = -(g^{\mu\nu}\partial_\mu\partial_\nu + a^\sigma\partial_\sigma + b), \quad (1)$$

where a and b are some matrix valued functions and $g^{\mu\nu}$ is the inverse metric tensor on the manifold. For a flat space $g^{\mu\nu} = \delta^{\mu\nu}$.

The heat kernel can be defined as follows:

$$K(t; x; y; L) = \langle x | \exp(-tL) | y \rangle = \sum_\lambda \phi_\lambda^\dagger(x) \phi_\lambda(y) \exp(-t\lambda), \quad (2)$$

where ϕ_λ is an eigenfunction of the operator L with the eigenvalue λ .

It satisfies the heat equation

$$(\partial_t + L_x)K(t; x; y; L) = 0 \quad (3)$$

with an initial condition

$$K(0; x; y; L) = \delta(x, y). \quad (4)$$

If we consider the fields in a finite volume then it is necessary to specify boundary conditions. Different choices are possible. In section 3.1 we will consider the case of periodic boundary conditions on imaginary time coordinate, which are specific for boson fields. In section 3.2 we will study bag boundary conditions imposed on fermion fields. If the normal to the boundary component of the fermion current $\psi^\dagger \gamma_n \psi$ vanishes at the boundary, one can impose bag boundary conditions, a particular case of mixed boundary conditions. We assume given two complementary projectors Π_\pm , $\Pi_- + \Pi_+ = I$ acting on a multi component field (the eigenfunction of the operator L) at each point of the boundary and define mixed boundary conditions by the relations

$$\Pi_- \psi|_{\partial M} = 0, \quad (\nabla_n + S) \Pi_+ \psi|_{\partial M} = 0, \quad (5)$$

where S is a matrix valued function on the boundary. In other words, the components $\Pi_- \psi$ satisfy Dirichlet boundary conditions, and $\Pi_+ \psi$ satisfy Robin (modified Neumann) ones.

It is convenient to define

$$\chi = \Pi_+ - \Pi_- . \quad (6)$$

Let $\{e_j\}$, $j = 1, \dots, n$ be a local orthonormal frame for the tangent space to the manifold and let on the boundary e_n be an inward pointing normal vector.

The extrinsic curvature is defined by the equation

$$L_{ab} = \Gamma_{ab}^n, \quad (7)$$

where Γ is the Christoffel symbol. For example, on the unit sphere S^{n-1} which bounds the unit ball in R^n the extrinsic curvature is $L_{ab} = \delta_{ab}$.

Curved space offers no complications in our approach compared to the flat case. Let $R_{\mu\nu\rho\sigma}$ be the Riemann tensor, and let $R_{\mu\nu} = R^\sigma_{\mu\nu\sigma}$ be the Ricci tensor. With our sign convention the scalar curvature $R = R^\mu_\mu$ is $+2$ on the unit sphere S^2 . In flat space the Riemann and Ricci tensors are equal to zero.

One can always introduce a connection ω_μ and another matrix valued function E so that L takes the form:

$$L = -(g^{\mu\nu} \nabla_\mu \nabla_\nu + E) \quad (8)$$

Here ∇_μ is a sum of covariant Riemannian derivative with respect to metric $g_{\mu\nu}$ and connection ω_μ . One can, of course, express E and ω in terms of a^μ , b and $g_{\mu\nu}$:

$$\omega_\mu = \frac{1}{2}g_{\mu\nu}(a^\nu + g^{\rho\sigma}\Gamma_{\rho\sigma}^\nu), \quad (9)$$

$$E = b - g^{\mu\nu}(\partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\rho\Gamma_{\mu\nu}^\rho) \quad (10)$$

For the future use we introduce also the field strength for ω :

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]. \quad (11)$$

The connection ω_μ will be used to construct covariant derivatives. The subscript $;\mu\dots\nu\sigma$ will be used to denote repeated covariant derivatives with the connection ω and the Christoffel connection on M . The subscript $:a\dots bc$ will denote repeated covariant derivatives containing ω and the Christoffel connection on the boundary. Difference between these two covariant derivatives is measured by the extrinsic curvature (7). For example, $E_{;ab} = E_{:ab} - L_{ab}E_{;n}$.

Let us define an integrated heat kernel for a hermitian operator L by the equation:

$$K(Q, L, t) := \text{Tr}(Q \exp(-tL)) = \int_M d^n x \sqrt{g} \text{tr}(Q(x)K(t; x; x; L)), \quad (12)$$

where $Q(x)$ is an hermitian matrix valued function, tr here is over matrix indices. For the boundary conditions we consider in this paper there exists an asymptotic expansion [26] as $t \rightarrow 0$:

$$K(Q, L, t) \simeq \sum_{k=0}^{\infty} a_k(Q, L) t^{(k-n)/2}. \quad (13)$$

According to the general theory [26] the coefficients $a_k(Q, L)$ are locally computable. This means that each $a_k(Q, L)$ can be represented as a sum of volume and boundary integrals of local invariants constructed from Q , Ω , E , the curvature tensor, and their derivatives. Boundary invariants may also include S , L_{ab} and χ . Total mass dimension of such invariants should be k for the volume terms and $k-1$ for the boundary ones.

At the moment several coefficients of the expansion (13) are known for the case of mixed boundary conditions (5) and matrix valued function Q (see

[9] for details of derivation; the formula (51) for a_4 was derived in [9] with additional restrictions $L_{ab} = 0$ and $S = 0$) :

$$a_0(Q, L) = (4\pi)^{-n/2} \int_M d^n x \sqrt{g} \operatorname{tr}(Q). \quad (14)$$

$$a_1(Q, L) = \frac{1}{4}(4\pi)^{-(n-1)/2} \int_{\partial M} d^{n-1} x \sqrt{h} \operatorname{tr}(\chi Q). \quad (15)$$

$$a_2(Q, L) = \frac{1}{6}(4\pi)^{-n/2} \left\{ \int_M d^n x \sqrt{g} \operatorname{tr}(6QE + QR) \right. \\ \left. + \int_{\partial M} d^{n-1} x \sqrt{h} \operatorname{tr}(2QL_{aa} + 12QS + 3\chi Q_{;n}) \right\}. \quad (16)$$

$$a_3(Q, L) = \frac{1}{384}(4\pi)^{-(n-1)/2} \int_{\partial M} d^{n-1} x \sqrt{h} \operatorname{tr} \{ Q(-24E + 24\chi E\chi \\ + 48\chi E + 48E\chi - 12\chi_{;a}\chi_{;a} + 12\chi_{;aa} - 6\chi_{;a}\chi_{;a}\chi + 16\chi R \\ + 8\chi R_{anan} + 192S^2 + 96L_{aa}S + (3 + 10\chi)L_{aa}L_{bb} \\ + (6 - 4\chi)L_{ab}L_{ab}) + Q_{;n}(96S + 192S^2) + 24\chi Q_{;nn} \}. \quad (17)$$

For a scalar function Q and mixed boundary conditions the coefficients a_4 and a_5 were already derived [27].

2.2 ζ -function

Zeta function of a positive operator L is defined by

$$\zeta_L(s) = \sum_{\lambda} \frac{1}{\lambda^s}, \quad (18)$$

where the sum is over all eigenvalues of the operator L . The zeta function is related to the heat kernel by the transformation

$$\zeta_L(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt t^{s-1} K(I, L, t). \quad (19)$$

Residues at the poles of the zeta function are related to the coefficients of the heat kernel expansion:

$$a_k(I, L) = \operatorname{Res}_{s=(n-k)/2}(\Gamma(s)\zeta_L(s)). \quad (20)$$

Here I is a unit matrix with a dimension of the matrix functions a^μ, b in (1). From (20) it follows that

$$a_n(I, L) = \zeta_L(0). \quad (21)$$

In Euclidean four dimensional space the zero temperature one-loop path integral over the boson fields $\phi = \sum_{\lambda} C_{\lambda} \phi_{\lambda}$ can be evaluated as follows (up to a normalization factor):

$$Z = \int d\phi e^{-\int d^4x \phi L \phi} \simeq \prod_{\lambda} \int \mu dC_{\lambda} e^{-\lambda C_{\lambda}^2} \simeq \mu^{\zeta_L(0)} \det L^{-1/2}. \quad (22)$$

Here we introduced the constant μ with a dimension of mass in order to keep a proper dimension of the measure in the functional integral. $\zeta_L(0)$ can be thought of as a number of eigenvalues of the operator L . For the operator L in the form (1) the number of eigenvalues is infinite, so $\zeta_L(0)$ yields a regularized value for this number.

The zero temperature one-loop effective action is defined then by

$$\begin{aligned} W &= -\ln Z = -\frac{1}{2} \ln \det L + \frac{1}{2} \zeta_L(0) \ln \mu^2 = \frac{1}{2} \zeta'_L(0) + \frac{1}{2} \zeta_L(0) \ln \mu^2 = \\ &= \frac{1}{2} \frac{\partial}{\partial s} (\mu^{2s} \zeta_L(s))|_{s=0} \end{aligned} \quad (23)$$

The term $\zeta_L(0) \ln \mu^2 = a_4(I, L) \ln \mu^2$ in the effective action W determines the one-loop beta function, this term describes renormalization of the one-loop logarithmic divergences appearing in the theory.

2.3 Free energy for boson fields

A finite temperature field theory is defined in Euclidean space, since for boson fields one has to impose periodic boundary conditions on imaginary time coordinate (antiperiodic boundary conditions for fermion fields respectively). A partition function is defined by

$$Z(\beta) = \text{Tr} e^{-\beta H}, \quad (24)$$

where H is a hamiltonian of the problem and $\beta = \hbar/T$. Let us choose the lagrangian density ρ in the form

$$\rho = -\frac{\partial^2}{\partial \tau^2} + L, \quad (25)$$

where τ is an imaginary time coordinate and L is a three dimensional spatial part of the density in the form (1). The free energy of the system is defined by

$$F(\beta) = -\frac{\hbar}{\beta} \ln Z(\beta) = -\frac{\hbar}{\beta} \ln \left(N_{\beta} \int D\phi \exp \left(- \int_0^{\beta} d\tau \int d^3x \phi \rho \phi \right) \right), \quad (26)$$

the integration is over all periodic fields satisfying $\phi(\tau + \beta) = \phi(\tau)$ (N_β is a normalization coefficient). As a result the eigenfunctions of ρ have the form $\exp(i\tau\omega_n)\phi_\lambda$, where $\omega_n = 2\pi n/\beta$ and $L\phi_\lambda = \lambda\phi_\lambda$. The free energy is thus equal to [22]

$$F = \frac{\hbar}{2\beta} \sum_{n=-\infty}^{+\infty} \sum_{\lambda} \ln \frac{(\omega_n^2 + \lambda)}{\mu^2} = -\frac{\hbar}{2\beta} \frac{\partial}{\partial s} (\mu^{2s} \zeta(s))|_{s=0}, \quad (27)$$

where we introduced ζ -function

$$\zeta(s) = \sum_{n=-\infty}^{+\infty} \sum_{\lambda} (\omega_n^2 + \lambda)^{-s} \quad (28)$$

and the parameter μ with a mass dimensionality in order to make the argument of the logarithm dimensionless (also see a previous section).

Then it is convenient to use the formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt t^{s-1} \sum_{n=-\infty}^{+\infty} \sum_{\lambda} e^{-t(\omega_n^2 + \lambda)}, \quad (29)$$

and separate $n = 0$ and other terms in the sum. For $n \neq 0$ terms we substitute the heat kernel expansion for the operator L at small t

$$\sum_{\lambda} e^{-\lambda t} = K(I; L; t) \simeq \sum_{k=0}^{\infty} a_k(I, L) t^{(k-3)/2} \quad (30)$$

and perform t integration, then we arrive at the high temperature expansion ($\beta \rightarrow 0$) for the free energy F :

$$\begin{aligned} F/\hbar = & -\frac{1}{2\beta} \zeta'_L(0) - \frac{1}{2\beta} \zeta_L(0) \ln(\mu^2) + (4\pi)^{3/2} \left[-\frac{a_0}{\beta^4} \frac{\pi^2}{90} - \frac{a_1}{\beta^3} \frac{\zeta_R(3)}{4\pi^{3/2}} - \frac{a_2}{\beta^2} \frac{1}{24} \right. \\ & + \frac{a_3}{\beta} \frac{1}{(4\pi)^{3/2}} \ln\left(\frac{\beta\mu}{2\pi}\right) - \frac{a_4}{16\pi^2} \left(\gamma + \ln \frac{\beta\mu}{2\pi} \right) \\ & \left. - \sum_{n \geq 5} \frac{a_n}{\beta^{4-n}} \frac{(2\pi)^{3/2-n}}{2\sqrt{2}} \Gamma\left(\frac{n-3}{2}\right) \zeta_R(n-3) \right]. \end{aligned} \quad (31)$$

Here $a_k \equiv a_k(I, L)$, $\zeta_R(s) = \sum_{n=1}^{+\infty} n^{-s}$ is a Riemann zeta function, $\zeta_L(s) = \sum_{\lambda} \lambda^{-s}$ is a zeta function of an operator L , γ is the Euler constant. The first two terms on the r.h.s. of (31) follow from the $n = 0$ term.

The term

$$-\frac{(4\pi)^{3/2} \hbar a_0}{\beta^4} \frac{\pi^2}{90} = -V \frac{\text{tr} I}{\hbar^3} \frac{\pi^2}{90} T^4 \quad (32)$$

is the leading high temperature contribution to the free energy.

The classical limit terms due to the equality $\zeta_L(0) = a_3$ can be rewritten as follows:

$$T\left(-\frac{1}{2}\zeta'_L(0) + \zeta_L(0) \ln \frac{\hbar}{2\pi T}\right) = T \sum_{\lambda} \ln \frac{\hbar\sqrt{\lambda}}{2\pi T}. \quad (33)$$

The terms on the l.h.s. of (33) yield a renormalized value of the terms on the r.h.s. of (33), since the sum on the righthandsight is generally divergent when the number of modes is infinite.

The term with a_4 determines the part of the free energy that appears due to one-loop logarithmic divergences and thus it depends on the dimensional parameter μ as in the zero temperature case.

2.4 Chiral anomaly in four dimensions for MIT bag boundary conditions

Consider the Dirac operator on an n -dimensional Riemannian manifold

$$\widehat{D} = \gamma^\mu \left(\partial_\mu + V_\mu + iA_\mu \gamma^5 - \frac{1}{8} [\gamma_\rho, \gamma_\sigma] \sigma_\mu^{[\rho\sigma]} \right) \quad (34)$$

in external vector V_μ and axial vector A_μ fields. We suppose that V_μ and A_μ are anti-hermitian matrices in the space of some representation of the gauge group. $\sigma_\mu^{[\rho\sigma]}$ is the spin-connection¹.

The Dirac operator transforms covariantly under infinitesimal local gauge transformations (the local gauge transformation is $\widehat{D} \rightarrow \exp(-\lambda)\widehat{D}\exp(\lambda)$):

$$\begin{aligned} \delta_\lambda A_\mu &= [A_\mu, \lambda] \\ \delta_\lambda V_\mu &= \partial_\mu \lambda + [V_\mu, \lambda] \\ \widehat{D} &\rightarrow \widehat{D} + [\widehat{D}, \lambda] \end{aligned} \quad (35)$$

and under infinitesimal local chiral transformations (the local chiral transformation is $\widehat{D} \rightarrow \exp(i\varphi\gamma_5)\widehat{D}\exp(i\varphi\gamma_5)$):

$$\begin{aligned} \tilde{\delta}_\varphi A_\mu &= \partial_\mu \varphi + [V_\mu, \varphi], \\ \tilde{\delta}_\varphi V_\mu &= -[A_\mu, \varphi], \\ \widehat{D} &\rightarrow \widehat{D} + i\{\widehat{D}, \gamma^5 \varphi\}. \end{aligned} \quad (36)$$

¹The spin-connection must be included even on a flat manifold if the coordinates are not Cartesian.

The parameters λ and φ are anti-hermitian matrices.

First we adopt the zeta-function regularization and write the one-loop effective action for Dirac fermions at zero temperature as ²

$$W = -\ln \det \widehat{D} = -\frac{1}{2} \ln \det \widehat{D}^2 = \frac{1}{2} \zeta'_{\widehat{D}^2}(0) + \frac{1}{2} \ln(\mu^2) \zeta_{\widehat{D}^2}(0), \quad (37)$$

where

$$\zeta_{\widehat{D}^2}(s) = \text{Tr}(\widehat{D}^{-2s}), \quad (38)$$

prime denotes differentiation with respect to s , and Tr is the functional trace.

The following identity holds:

$$\zeta_A(s) = \text{Tr} A^{-s} \Rightarrow \delta \zeta_A(s) = -s \text{Tr}((\delta A) A^{-s-1}). \quad (39)$$

Due to the identity (39)

$$\delta_\lambda \zeta_{\widehat{D}^2}(s) = -\left(2s \text{Tr}([\widehat{D}, \lambda] \widehat{D}^{-2s-1})\right) = -2s \left(\text{Tr}([\widehat{D}^{-2s}, \lambda])\right) = 0, \quad (40)$$

so the effective action (37) is gauge invariant, $\delta_\lambda W = 0$.

The chiral anomaly is by definition equal to the variation of W under an infinitesimal chiral transformation. Using (39) we obtain:

$$\tilde{\delta}_\varphi \zeta_{\widehat{D}^2}(s) = -\left(2is \text{Tr}(\{\widehat{D}, \gamma^5 \varphi\} \widehat{D}^{-2s-1})\right) = -4is \left(\text{Tr}(\gamma^5 \varphi \widehat{D}^{-2s})\right), \quad (41)$$

and the anomaly reads

$$\mathcal{A} := \tilde{\delta}_\varphi W = \frac{1}{2} \tilde{\delta}_\varphi \zeta'_{\widehat{D}^2}(0) = -2 \text{Tr}(i\gamma^5 \varphi \widehat{D}^{-2s})|_{s=0}. \quad (42)$$

The heat kernel is related to the zeta function by the Mellin transformation:

$$\text{Tr}(i\gamma^5 \varphi \widehat{D}^{-2s}) = \Gamma(s)^{-1} \int_0^\infty dt t^{s-1} K(i\gamma^5 \varphi, \widehat{D}^2, t). \quad (43)$$

In particular, after the substitution of the heat kernel expansion (13) into the formula (43) we obtain

$$\mathcal{A} = -2a_n(i\gamma^5 \varphi, \widehat{D}^2). \quad (44)$$

The same expression for the anomaly follows also from the Fujikawa approach [7].

²The one-loop effective action is proportional to Planck constant \hbar , in what following we put $\hbar = 1$.

One can also derive the expression for the anomaly (44) from Schwinger's effective action. One should start from an identity:

$$\ln \lambda = - \int_0^{+\infty} \frac{dt}{t} e^{-t\lambda} \quad (45)$$

Then the change in the effective action due to chiral transformations can be written:

$$\begin{aligned} \mathcal{A} &= \tilde{\delta}_\varphi \left(-\frac{1}{2} \ln \det \hat{D}^2 \right) = \tilde{\delta}_\varphi \text{tr} \int_0^{+\infty} \frac{dt}{2t} e^{-t\hat{D}^2} = \\ &= -i \text{tr} \int_0^{+\infty} \{ \gamma^5 \varphi, \hat{D}^2 \} e^{-t\hat{D}^2} = 2i \text{tr} \gamma^5 \varphi \int_0^{+\infty} \frac{\partial}{\partial t} e^{-t\hat{D}^2} = \\ &= -2 \lim_{t \rightarrow 0} \text{tr} i \gamma^5 \varphi e^{-t\hat{D}^2} = -2a_n(i\gamma^5 \varphi, \hat{D}^2). \end{aligned} \quad (46)$$

We impose local boundary conditions:

$$\Pi_- \psi|_{\partial M} = 0, \quad \Pi_- = \frac{1}{2} (1 - \gamma^5 \gamma_n), \quad (47)$$

which are nothing else than a Euclidean version of the MIT bag boundary conditions [25]. For these boundary conditions $\Pi_-^\dagger = \Pi_-$, and the normal component of the fermion current $\psi^\dagger \gamma_n \psi$ vanishes on the boundary. Spectral properties of the Dirac operator for bag boundary conditions are intensively studied [28].

Since \hat{D} is a first order differential operator it was enough to fix the boundary conditions (47) on a half of the components. To proceed with a second order operator $L = \hat{D}^2$ we need boundary conditions on the remaining components as well. They are defined by the consistency condition [29]:

$$\Pi_- \hat{D} \psi|_{\partial M} = 0, \quad (48)$$

which is equivalent to the Robin boundary condition

$$(\nabla_n + S) \Pi_+ \psi|_{\partial M} = 0, \quad \Pi_+ = \frac{1}{2} (1 + \gamma^5 \gamma_n) \quad (49)$$

with

$$S = -\frac{1}{2} \Pi_+ L_{aa}. \quad (50)$$

In the paper [9] the following expression for a coefficient $a_4(Q, L)$ with an hermitian matrix valued function Q and conditions (5), $L_{ab} = 0$ (flat

boundaries), $S = 0$ was obtained:

$$\begin{aligned}
a_4(Q, L) = & \frac{1}{360} (4\pi)^{-n/2} \left\{ \int_M d^n x \sqrt{g} \operatorname{tr} \left\{ Q(60E_{;\mu}{}^\mu + 60RE + 180E^2 \right. \right. \\
& + 30\Omega_{\mu\nu}\Omega^{\mu\nu} + 12R_{;\mu}{}^\mu + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \Big\} \\
& + \int_{\partial M} d^{n-1}x \sqrt{h} \operatorname{tr} \left\{ Q\{30E_{;n} + 30\chi E_{;n}\chi + 90\chi E_{;n} + 90E_{;n}\chi \right. \\
& + 18\chi\chi_{;a}\Omega_{an} + 12\chi_{;a}\Omega_{an}\chi + 18\Omega_{an}\chi\chi_{;a} - 12\chi\Omega_{an}\chi_{;a} \\
& + 6[\chi\Omega_{an}\chi, \chi_{;a}] + 54[\chi_{;a}, \Omega_{an}] + 30[\chi, \Omega_{an;a}] + 12R_{;n} + 30\chi R_{;n} \Big\} + \\
& + Q_{;n}(-30E + 30\chi E\chi + 90\chi E + 90E\chi - \\
& \left. - 18\chi_{;a}\chi_{;a} + 30\chi_{;aa} - 6\chi_{;a}\chi_{;a}\chi + 30\chi R) + 30\chi Q_{;\mu}{}^{\mu n} \right\}. \quad (51)
\end{aligned}$$

To obtain the chiral anomaly in four dimensions³ with MIT bag boundary conditions one has to calculate the coefficient $a_4(Q, L)$ (51) with $L = \widehat{D}^2$, $Q = i\gamma^5\phi$ and substitute it into (44). We define $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]$, $A_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$, $D_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\mu\nu}^\rho A_\rho + [V_\mu, A_\nu]$. The anomaly contains two contributions:

$$\mathcal{A} = \mathcal{A}_V + \mathcal{A}_b. \quad (52)$$

In the volume part

$$\begin{aligned}
\mathcal{A}_V = & \frac{-1}{180(2\pi)^2} \int_M d^4x \sqrt{g} \operatorname{tr} \varphi \left(-120 [D_\mu V^{\mu\nu}, A_\nu] \right. \\
& + 60 [D_\mu A_\nu, V^{\mu\nu}] - 60 D_\mu D^\mu D_\nu A^\nu + 120 \{ \{ D_\mu A_\nu, A^\nu \}, A^\mu \} \\
& + 60 \{ D_\mu A^\mu, A_\nu A^\nu \} + 120 A_\mu D_\nu A^\nu A^\mu + 30 [[A_\mu, A_\nu], A^{\mu\nu}] \\
& + \epsilon_{\mu\nu\rho\sigma} \{ -45 i V^{\mu\nu} V^{\rho\sigma} + 15 i A^{\mu\nu} A^{\rho\sigma} - 30 i (V^{\mu\nu} A^\rho A^\sigma + A^\mu A^\nu V^{\rho\sigma}) \\
& - 120 i A^\mu V^{\nu\rho} A^\sigma + 60 i A^\mu A^\nu A^\rho A^\sigma \} - 60 (D_\sigma A_\nu) R^{\nu\sigma} + 30 (D_\mu A^\mu) R \\
& \left. - \frac{15i}{8} \epsilon_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\eta\theta} R^{\rho\sigma\eta\theta} \right) \quad (53)
\end{aligned}$$

only the $DA - R$ terms seem to be new [9] (for flat space it can be found e.g. in [8]).

The boundary part

$$\begin{aligned}
\mathcal{A}_b = & \frac{-1}{180(2\pi)^2} \int_{\partial M} d^3x \sqrt{h} \operatorname{tr} \left(12 i \epsilon^{abc} \{ A_b, \varphi \} D_a A_c \right. \\
& + 24 \{ \varphi, A^a \} \{ A_a, A_n \} - 60 [A^a, \varphi] (V_{na} - [A_n, A_a]) \\
& \left. + 60 (D_n \varphi) D_\mu A^\mu \right) \quad (54)
\end{aligned}$$

³In two dimensions ($n = 2$) the boundary part of the chiral anomaly with MIT bag boundary conditions is equal to zero [9].

is new [9]. It has been derived under the two restrictions: $S = 0$ and $L_{ab} = 0$. Note, that in the present context, the first condition ($S = 0$) actually follows from the second one ($L_{ab} = 0$) due to (50).

3 Casimir effect for rectangular cavities

3.1 Casimir energy of two perfectly conducting parallel plates

The Casimir energy is usually defined as

$$E = \sum_i \frac{\hbar \omega_i}{2}, \quad (55)$$

where the sum is over all eigenfrequencies of the system. In what following we put $\hbar = 1$. We start from the well known case of two perfectly conducting plates separated by a distance a from each other. In this case the eigenfrequencies ω_i are defined as follows:

$$\omega_{TE} = \sqrt{(\pi n/a)^2 + k_x^2 + k_y^2}, \quad n = 1.. + \infty \quad (56)$$

$$\omega_{TM} = \sqrt{(\pi n/a)^2 + k_x^2 + k_y^2}, \quad n = 1.. + \infty \quad (57)$$

$$\omega_{\text{mainwave}} = \sqrt{k_x^2 + k_y^2}, \quad (58)$$

so that the Casimir energy can be written as

$$E = \frac{S}{2} \left(\sum_{n=1}^{+\infty} + \sum_{n=0}^{+\infty} \right) \iint_{-\infty}^{+\infty} \frac{dk_x dk_y}{(2\pi)^2} \sqrt{(\pi n/a)^2 + k_x^2 + k_y^2}, \quad (59)$$

S is the surface of each plate. The first sum is equivalent to the sum over eigenfrequencies of the scalar field satisfying Dirichlet boundary conditions, the second sum is equivalent to the sum over eigenfrequencies of the scalar field satisfying Neumann boundary conditions.

The expression for the Casimir energy written in this form is divergent. One has to regularize it somehow to obtain a finite answer for the energy. Different methods were used for this purpose. In the present paper we suggest a method which makes calculations of determinants straightforward and easy to perform.

By making use of an identity

$$\int_{-\infty}^{+\infty} \frac{ds}{2\pi} \ln \frac{s^2 + k^2}{s^2} = k \quad (60)$$

we can see that up to an irrelevant constant the Casimir energy can be written in the form (we introduce a dimensional parameter μ by the same reasoning as in (22) or (27)):

$$\begin{aligned}
E &= \frac{S}{2} \sum_{n=-\infty}^{+\infty} \iiint_{-\infty}^{+\infty} \frac{dk_x dk_y ds}{(2\pi)^3} \ln \left(\frac{(\pi n/a)^2 + k_x^2 + k_y^2 + s^2}{\mu^2} \right) = \\
&= \frac{S}{(2\pi)^2} \left(\sum_{n=1}^{+\infty} + \sum_{n=0}^{+\infty} \right) \int_0^{+\infty} dk \, k^2 \ln \left(n^2 + (ka/\pi)^2 \right) - \\
&\quad - \frac{1}{T} \ln(a\mu/\pi) a_4(I, L).
\end{aligned} \tag{61}$$

Now the expression for the Casimir energy is written in the standard $\text{Tr} \ln = \ln \text{Det}$ form, which is usual for one-loop effective actions in quantum field theory. The coefficient $a_4(I, L)$ is equal to zero for our current choice of the operator L and boundary geometry.

At this point we introduce a regularization - we restrict integrations over momenta by some cut off K in the momentum space. The sums over n are also restricted as follows:

$$\sum_{n=1}^{+\infty} + \sum_{n=0}^{+\infty} \rightarrow \sum_{n=1}^{+N} + \sum_{n=0}^{+N}. \tag{62}$$

The regularized Casimir energy is defined by:

$$E_{reg} = \frac{S}{(2\pi)^2} \left(\sum_{n=1}^{+N} + \sum_{n=0}^{+N} \right) \int_0^{+K} dk \, k^2 \ln \left(n^2 + (ka/\pi)^2 \right). \tag{63}$$

It is convenient to perform a summation over n first. The following identity holds:

$$\sum_{n=1}^N \ln(n^2 + (ka/\pi)^2) = \sum_{n=1}^N \ln(1 + (ka)^2/\pi^2 n^2) + \sum_{n=1}^N \ln(n^2). \tag{64}$$

The first sum in (64) can be calculated in the $N \rightarrow +\infty$ limit by use of an identity:

$$\prod_{n=1}^{+\infty} \left(1 + (ka)^2/\pi^2 n^2 \right) = \frac{\sinh(ka)}{ka}. \tag{65}$$

The second sum in (64) can be derived by use of a Stirling formula (which is exact in the large N limit):

$$N! \sim \sqrt{2\pi} N^{N+1/2} e^{-N}, \tag{66}$$

so in the large N limit it is possible to write:

$$\sum_{n=1}^N \ln(n^2) = 2 \ln N! = \ln(2\pi) + f(N). \quad (67)$$

In the large N limit the Dirichlet sum (64) can be rewritten as:

$$\sum_{n=1}^N \ln(n^2 + (ka/\pi)^2) = ka + \ln(1 - \exp(-2ka)) - \ln(ka/\pi) + f(N). \quad (68)$$

The sum over Neumann modes can be rewritten as follows:

$$\sum_{n=0}^N \ln(n^2 + (ka/\pi)^2) = ka + \ln(1 - \exp(-2ka)) + \ln(ka/\pi) + f(N). \quad (69)$$

It is possible to add any finite number that does not depend on a to the regularized Casimir energy E_{reg} (63) (the force between the plates is being measured in experiments, so the energy can be defined up to a constant). We add the surface term

$$-\frac{S}{(2\pi)^2} \int_0^K dk \, k^2 \, 2f(N) \quad (70)$$

to the regularized Casimir energy E_{reg} (63). Doing so we obtain

$$E_{reg} = 2 \frac{Sa}{(2\pi)^2} \int_0^K dk \, k^3 + \frac{S}{(2\pi)^2} \int_0^K dk \, k^2 \, 2 \ln(1 - \exp(-2ka)) \quad (71)$$

The first term in (71) is twice the regularized Casimir energy of the free scalar field since it can be rewritten as

$$2 \frac{V}{(2\pi)^3} \int_0^K dk \, 4\pi k^2 \, \frac{k}{2}. \quad (72)$$

This term should be subtracted because we are interested in the change of the ground state energy when the plates are inserted into the free space.

Next we perform the limits $K \rightarrow +\infty, N \rightarrow +\infty$. The Casimir energy is thus

$$E = \frac{S}{(2\pi)^2} \int_0^{+\infty} dk \, k^2 \, 2 \ln(1 - \exp(-2ka)) = -\frac{S\pi^2}{720a^3}, \quad (73)$$

which is the well known result by Casimir [10].

After elaborations we summarize the key points of the method, which is valid for the calculations in cylindrical cavities with arbitrary cross sections.

Suppose we want to calculate $\text{Tr} \ln$ of the second order operator $L^{(4)} = L^{(1)} + L^{(3)}$, where the dimensionalities of the operators are denoted by numbers. At zero temperature in our case of interest the operator $L^{(3)}$ describes a scalar field inside an infinite waveguide of an arbitrary cross section with Dirichlet or Neumann boundary conditions imposed. The eigenmodes of the operators $L^{(1)}$ and $L^{(3)}$ are denoted by $\lambda_i^{(1)}$ and $\lambda_k^{(3)}$ respectively. The following expression is finite (as can be seen from the heat kernel expansion):

$$\frac{1}{2} \ln \prod_i \frac{\lambda_i^{(1)} + \lambda_k^{(3)}}{\lambda_i^{(1)}} \equiv \frac{1}{2} \ln \prod_i \left(1 + \frac{\lambda_k^{(3)}}{\lambda_i^{(1)}} \right). \quad (74)$$

To obtain the initial determinant one should add to (74) the term

$$\frac{1}{2} \ln \prod_i^N \lambda_i^{(1)} = s(N) + \text{const} \quad (75)$$

The sum of (74) and (75) generally has the following structure (to obtain the total Casimir energy the sum over indices k and p has to be performed):

$$\frac{a \sqrt{\lambda_k^{(3)}}}{2} + g(\lambda_k^{(3)} a^2) + h_{Dir}^{surf}(\lambda_p^{(2)}/\mu^2) + h_{Neum}^{surf}(\lambda_p^{(2)}/\mu^2) + 2s(N), \quad (76)$$

where

$$\sum_k g(\lambda_k^{(3)} a^2) - \text{a convergent sum}, \quad (77)$$

the term (77) yields the energy of interaction for two flat parallel plates separated by a distance a inside an infinite waveguide of the same cross section as these parallel plates (the walls of a perfectly conducting waveguide are perpendicular to two flat parallel perfectly conducting plates inside it). The term (77) yields an experimentally measurable contribution to the Casimir energy of the cavity (see Sec. 3.7 for details).

The term $\sum_k a \sqrt{\lambda_k^{(3)}}/2$ is equal to the self-energy of an infinite waveguide when $a \rightarrow \infty$. For rectangular cavities the term $\sum_k a \sqrt{\lambda_k^{(3)}}/2$ can be transformed to $\text{Tr} \ln L_2^{(4)}$ in the same manner as in the beginning of this section (see a transition from (59) to (61)). For the operator $L_2^{(4)}$ we repeat the step (74) and continue this cycle until the first term in the righthandsight of (76) gets the form of the vacuum energy in an infinite space, i.e. the form (72).

The h^{surf} terms describe the self-energies of two parallel plates inside the waveguide due to Dirichlet and Neumann modes, these self-energies do

not depend on a . For flat boundaries Dirichlet and Neumann boundary contributions to the Casimir energy cancel each other identically as can be seen from the Seeley coefficient a_1 (15), for two parallel plates it can be seen from the expressions (68) and (69).

A contribution from the last term $s(N)$ is proportional to $a_3(I, L^{(3)})$ (at zero temperature it is just the effective number of modes inside an infinite perfectly conducting waveguide, and thus it is not relevant to the energy of interaction between the two plates inside the waveguide) and $a_4(I, L^{(4)})$ (this term is also not relevant when the interaction of the two plates inside an infinite waveguide is studied).

To implement (74) we used the following equality:

$$\prod_{n=1}^{+\infty} \frac{(\pi n/a)^2 + \lambda_k^{(3)}}{(\pi n/a)^2} \equiv \prod_{n=1}^{+\infty} \left(1 + \frac{\lambda_k^{(3)} a^2}{\pi^2 n^2}\right) = \frac{\sinh a \sqrt{\lambda_k^{(3)}}}{a \sqrt{\lambda_k^{(3)}}}. \quad (78)$$

3.2 Casimir energy of a perfectly conducting rectangular waveguide

For a perfectly conducting rectangular waveguide the technical issues can be done in analogy with two parallel plates. We tacitly assume that the reader understood how the regularization is introduced in our method, so we will write only main steps without bothering too much on divergent form of some expressions. The Casimir energy for unit length is:

$$E = \frac{1}{2} \sum_{n_1} \sum_{n_2} \iint_{-\infty}^{+\infty} \frac{dp_1 dp_2}{(2\pi)^2} \ln \left((\pi n_1/a)^2 + (\pi n_2/b)^2 + p_1^2 + p_2^2 \right) \quad (79)$$

For TM modes n_1 and n_2 take positive integer values from 1 to $+\infty$, for TE modes n_1 and n_2 take positive integer values and one of them can be equal to zero ($n_1 = n_2 = 0$ corresponds to the main wave case).

So the energy can be rewritten as:

$$\begin{aligned} E &= \frac{1}{4} \sum_{n_1, n_2=-\infty}^{+\infty} \iint \frac{dp_1 dp_2}{(2\pi)^2} \ln \left((\pi n_1/a)^2 + (\pi n_2/b)^2 + p_1^2 + p_2^2 \right) = \\ &= \frac{1}{4ab} \sum_{n_1, n_2=-\infty}^{+\infty} \iint \frac{dp_1 dp_2}{(2\pi)^2} \ln \left((\pi n_1)^2 t + (\pi n_2)^2 \frac{1}{t} + p_1^2 + p_2^2 \right), \end{aligned} \quad (80)$$

where $t = b/a$ or $t = a/b$. Using formula (78) for multiplication over n_1 we

obtain for the energy:

$$E = \frac{1}{4ab} \sum_{n_2=-\infty}^{+\infty} \iint \frac{dp_1 dp_2}{(2\pi)^2} 2 \ln \sinh \sqrt{(\pi n_2/t)^2 + \frac{p_1^2 + p_2^2}{t}} = \quad (81)$$

$$= \frac{1}{4ab} \sum_{n_2=-\infty}^{+\infty} \iint \frac{dp_1 dp_2}{(2\pi)^2} \left[2 \sqrt{(\pi n_2/t)^2 + \frac{p_1^2 + p_2^2}{t}} - (2 \ln 2) + \quad (82)$$

$$+ 2 \ln \left(1 - \exp \left(-2 \sqrt{(\pi n_2/t)^2 + \frac{p_1^2 + p_2^2}{t}} \right) \right) \right]. \quad (83)$$

A contribution from the term $(-2 \ln 2)$ in (82) should be subtracted following the analysis of Section 3.1. The part (83) with the logarithm is finite, it contributes to the finite final answer for the Casimir energy.

For the first term in (82) we get:

$$\frac{1}{2ab} \sum_{n_2=-\infty}^{+\infty} \iint \frac{dp_1 dp_2}{(2\pi)^2} \sqrt{(\pi n_2/t)^2 + \frac{p_1^2 + p_2^2}{t}} = \quad (84)$$

$$= \frac{1}{2t^2 ab} \sum_{n_2=-\infty}^{+\infty} \iiint \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \ln \left((\pi n_2)^2 + p_1^2 + p_2^2 + p_3^2 \right) = \quad (85)$$

$$= -\frac{\pi^2}{720t^2 ab} \quad (86)$$

because up to a numerical coefficient the expression (85) is just the same as the formula (61).

So the Casimir energy for unit length of a rectangular waveguide can be written as the sum of (83) and (86) :

$$\begin{aligned} E_{\text{waveguide}}(a, b) &= \\ &= -\frac{\pi^2}{720t^2 ab} + \frac{t}{4\pi ab} \sum_{n=-\infty}^{+\infty} \int_0^{+\infty} dp \, p \, \ln \left(1 - \exp \left(-2 \sqrt{\frac{\pi^2 n^2}{t^2} + p^2} \right) \right) \end{aligned} \quad (87)$$

3.3 Casimir energy of a perfectly conducting rectangular cavity

The Casimir energy in this case can be written as:

$$\begin{aligned}
E &= \sum_{n_1, n_2, n_3=1}^{+\infty} \sqrt{\left(\frac{\pi n_1}{a}\right)^2 + \left(\frac{\pi n_2}{b}\right)^2 + \left(\frac{\pi n_3}{c}\right)^2} + \\
&+ \frac{1}{2} \sum_{n_1, n_2=1}^{+\infty} \sqrt{\left(\frac{\pi n_1}{a}\right)^2 + \left(\frac{\pi n_2}{b}\right)^2} + \\
&+ \frac{1}{2} \sum_{n_1, n_3=1}^{+\infty} \sqrt{\left(\frac{\pi n_1}{a}\right)^2 + \left(\frac{\pi n_3}{c}\right)^2} + \\
&+ \frac{1}{2} \sum_{n_2, n_3=1}^{+\infty} \sqrt{\left(\frac{\pi n_2}{b}\right)^2 + \left(\frac{\pi n_3}{c}\right)^2} = \\
&= \frac{1}{8} \sum_{n_1, n_2, n_3=-\infty}^{+\infty} \sqrt{\left(\frac{\pi n_1}{a}\right)^2 + \left(\frac{\pi n_2}{b}\right)^2 + \left(\frac{\pi n_3}{c}\right)^2} - \\
&- \sum_{n_1=-\infty}^{+\infty} \frac{1}{8} \sqrt{\left(\frac{\pi n_1}{a}\right)^2} - \sum_{n_2=-\infty}^{+\infty} \frac{1}{8} \sqrt{\left(\frac{\pi n_2}{b}\right)^2} - \sum_{n_3=-\infty}^{+\infty} \frac{1}{8} \sqrt{\left(\frac{\pi n_3}{c}\right)^2}
\end{aligned} \tag{88}$$

Using formula (78) and technique described in previous subsections we obtain:

$$\begin{aligned}
&\frac{1}{8} \sum_{n_1, n_2, n_3=-\infty}^{+\infty} \sqrt{\left(\frac{\pi n_1}{a}\right)^2 + \left(\frac{\pi n_2}{b}\right)^2 + \left(\frac{\pi n_3}{c}\right)^2} = a E_{\text{waveguide}}(b, c) + \\
&+ \frac{1}{4} \sum_{n_2, n_3=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \ln \left(1 - \exp \left(-2a \sqrt{\left(\frac{\pi n_2}{b}\right)^2 + \left(\frac{\pi n_3}{c}\right)^2 + p^2} \right) \right)
\end{aligned} \tag{89}$$

The remaining terms should be calculated (using formula (78) again) as follows:

$$\begin{aligned}
\sum_{n_1=-\infty}^{+\infty} \frac{1}{8} \sqrt{\left(\frac{\pi n_1}{a}\right)^2} &= \frac{1}{8} \sum_{n_1=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \ln \left(\left(\frac{\pi n_1}{a}\right)^2 + p^2 \right) = \\
&= \frac{1}{4} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \ln \left(1 - \exp(-2ap) \right) = -\frac{\pi}{48a}
\end{aligned} \tag{90}$$

As a result for the Casimir energy of the cavity we obtain:

$$E_{cavity}(a, b, c) = \frac{\pi}{48a} + \frac{\pi}{48b} + \frac{\pi}{48c} + aE_{waveguide}(b, c) + \frac{1}{4} \sum_{n_2, n_3=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \ln \left(1 - \exp \left[-2a \sqrt{\left(\frac{\pi n_2}{b} \right)^2 + \left(\frac{\pi n_3}{c} \right)^2 + p^2} \right] \right). \quad (91)$$

3.4 Relation to the argument principle

An argument principle is a convenient method of summation over the eigenmodes of the system (see [30] and [31] for its applications). The argument principle states:

$$\frac{1}{2\pi i} \oint \phi(\omega) \frac{d}{d\omega} \ln f(\omega) d\omega = \sum \phi(\omega_0) - \sum \phi(\omega_\infty), \quad (92)$$

where ω_0 are zeroes and ω_∞ are poles of the function $f(\omega)$ inside the contour of integration. For the Casimir energy $\phi(\omega) = \omega/2$. We choose

$$f(\omega) = \frac{2 \sin \left[a \sqrt{\omega^2 - k_x^2 - k_y^2} \right]}{\mu \sqrt{\omega^2 - k_x^2 - k_y^2}} \quad (93)$$

in case of a scalar field satisfying Dirichlet boundary conditions on the plates. The contour lies on an imaginary axis, a contribution from the right semicircle with a large radius is negligible. A denominator is chosen in this form to remove $\omega^2 = k_x^2 + k_y^2$ from the roots of the equation $f(\omega) = 0$. In this case

we proceed as follows:

$$\begin{aligned}
E_{Dir} &= \frac{-S}{2\pi i} \iint \frac{dk_x dk_y}{(2\pi)^2} \int_{-i\infty}^{+i\infty} d\omega \frac{\omega}{2} \frac{\partial}{\partial \omega} \ln \frac{2 \sin \left[a \sqrt{\omega^2 - k_x^2 - k_y^2} \right]}{\mu \sqrt{\omega^2 - k_x^2 - k_y^2}} = \\
&= \frac{S}{4\pi i} \iint \frac{dk_x dk_y}{(2\pi)^2} \int_{-i\infty}^{+i\infty} d\omega \ln \frac{2 \sin \left[a \sqrt{\omega^2 - k_x^2 - k_y^2} \right]}{\mu \sqrt{\omega^2 - k_x^2 - k_y^2}} = \\
&= \frac{S}{2} \iiint \frac{dk_x dk_y d\tilde{\omega}}{(2\pi)^3} \ln \frac{2 \sinh \left[a \sqrt{\tilde{\omega}^2 + k_x^2 + k_y^2} \right]}{\mu \sqrt{\tilde{\omega}^2 + k_x^2 + k_y^2}} = \\
&= \frac{S}{(2\pi)^2} \int_0^{+\infty} dk k^2 \ln \left(1 - \exp(-2ka) \right) + \\
&\quad + \frac{V}{(2\pi)^3} \int_0^K dk 4\pi k^2 \frac{k}{2} - \frac{S}{(2\pi)^2} \int_0^K dk k^2 \ln(\mu k) = \\
&= -\frac{S\pi^2}{1440a^3} + \text{volume free space contribution} + \\
&\quad + \text{surface contribution.}
\end{aligned} \tag{94}$$

Here $\omega = i\tilde{\omega}$. We see that the argument principle is in agreement with (78).

3.5 Zeta function regularization for the cavity

ζ -function has already been discussed in this paper, so it is natural to describe regularization of the Casimir energy for the cavity in terms of ζ -function. Usually the Casimir energy is regularized as follows:

$$E = \frac{1}{2} \sum_{\omega_l} \omega_l^{-s}, \tag{95}$$

where s is large enough to make (95) convergent. Then we should continue analytically (95) to the value $s = -1$, this procedure yields the renormalized finite Casimir energy. In our case eigenfrequencies ω_l should be taken from (88). So the regularized Casimir energy of the cavity $E_{cavity}(a, b, c, s)$ can be written in terms of Epstein $Z_3(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}; s)$ and Riemann $\zeta_R(s)$ zeta functions:

$$\begin{aligned}
E_{cavity}(a, b, c, s) &= \frac{\pi}{8} \left(\sum_{n_1, n_2, n_3 = -\infty}^{+\infty} \left[\left(\frac{n_1}{a} \right)^2 + \left(\frac{n_2}{b} \right)^2 + \left(\frac{n_3}{c} \right)^2 \right]^{-s/2} - \right. \\
&\quad \left. - 2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \sum_{n=1}^{+\infty} \frac{1}{n^s} \right)
\end{aligned} \tag{96}$$

$$Z_3\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}; s\right) = \sum_{n_1, n_2, n_3=-\infty}^{+\infty'} \left[\left(\frac{n_1}{a}\right)^2 + \left(\frac{n_2}{b}\right)^2 + \left(\frac{n_3}{c}\right)^2 \right]^{-s/2} \quad (97)$$

$$\zeta_R(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (98)$$

The prime means that the term with all $n_i = 0$ should be excluded from the sum. The reflection formulas for an analytical continuation of zeta functions exist:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta_R(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{(s-1)/2} \zeta_R(1-s) \quad (99)$$

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} Z_3(a, b, c; s) = (abc)^{-1} \Gamma\left(\frac{3-s}{2}\right) \pi^{(s-3)/2} Z_3\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}; 3-s\right) \quad (100)$$

By use of reflection formulas (99), (100) one gets:

$$Z_3\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}; -1\right) = -\frac{abc}{2\pi^3} Z_3(a, b, c; 4) \quad (101)$$

$$\zeta_R(-1) = -\frac{1}{12}. \quad (102)$$

The renormalized Casimir energy can therefore be written as:

$$E_{cavity}(a, b, c) = -\frac{abc}{16\pi^2} Z_3(a, b, c; 4) + \frac{\pi}{48} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right). \quad (103)$$

One can check that the formulas (103) and (91) coincide identically and yield the Casimir energy for a perfectly conducting cavity.

3.6 Geometric interpretation

In this section we suggest a geometric interpretation of the main formulas in terms of geometric optics. This interpretation clarifies the physical meaning of the results (87), (91) obtained, which is always important for further generalizations in more complicated cases.

Several geometric approaches - a semiclassical method [32], a worldline approach [33] and a method of geometric optics [34] have been introduced recently for the evaluation of the Casimir energies. Our formulas (87), (91) yield a simple geometric interpretation for the Casimir energy of the rectangular cavities in terms of geometric optics.

Optical contributions to the Green's function of the scalar field with Dirichlet and Neumann boundary conditions have the form:

$$G_{optical}^D(\mathbf{x}, \mathbf{x}', \omega_i) = \frac{1}{4\pi} \sum_n (-1)^n \sqrt{\Delta_n(\mathbf{x}, \mathbf{x}')} \exp(i\omega_i l_n(\mathbf{x}, \mathbf{x}')) \quad (104)$$

$$G_{optical}^N(\mathbf{x}, \mathbf{x}', \omega_i) = \frac{1}{4\pi} \sum_n \sqrt{\Delta_n(\mathbf{x}, \mathbf{x}')} \exp(i\omega_i l_n(\mathbf{x}, \mathbf{x}')). \quad (105)$$

Here $l_n(\mathbf{x}, \mathbf{x}')$ is the length of the optical path that starts from \mathbf{x} and arrives at \mathbf{x}' after n reflections from the boundary. $\Delta_n(\mathbf{x}, \mathbf{x}')$ is the enlargement factor of classical ray optics. For planar boundaries it is given by $\Delta_n(\mathbf{x}, \mathbf{x}') = 1/l_n^2$. From (73) it follows that for two parallel plates the Casimir energy of the electromagnetic field can be expanded as:

$$\begin{aligned} E &= \frac{S}{2\pi^2} \int_0^{+\infty} dk \, k^2 \ln(1 - \exp(-2ka)) = \\ &= -2aS \int_0^{+\infty} dk \frac{4\pi k^2}{(2\pi)^3} \sum_{n=1}^{+\infty} \frac{\exp(-2ank)}{2an} = - \sum_{\omega_i} \sum_{n=1}^{+\infty} \frac{\exp(-2an\omega_i)}{2an}. \end{aligned} \quad (106)$$

Here \sum_{ω_i} is a sum over all photon states (with frequencies $\omega_i = \sqrt{k_x^2 + k_y^2 + k_z^2}$) in an infinite space. The righthandsight of (106) can be written in terms of optical Green's functions:

$$\sum_{\omega_i} \sum_{n=1}^{+\infty} \frac{\exp(-2an\omega_i)}{2an} = 2\pi \sum_{\omega_i} (G_{optical}^D(\mathbf{x}, \mathbf{x}, a, i\omega_i) + G_{optical}^N(\mathbf{x}, \mathbf{x}, a, i\omega_i)) \quad (107)$$

Note that terms with odd reflections from Dirichlet and Neumann Green's functions cancel each other due to the factor $(-1)^n$ present in optical Dirichlet Green's function. This is why only periodic paths with even number of reflections from the boundary $l_{2n} = 2an$ enter into the expression for the Casimir energy.

Now consider the formula for the cavity (91) (to obtain the Casimir energy of a waveguide (87) in terms of optical Green's functions the arguments are the same, just start from two parallel plates). Imagine that there is a waveguide with side lengths b and c . In order to obtain the rectangular cavity we have to insert two perfectly conducting plates with side lengths b and c (and a distance a apart) inside the waveguide. The eigenfrequencies that existed in a waveguide were equal to $\omega_{wave} = \sqrt{(\pi n_2/b)^2 + (\pi n_3/c)^2 + p^2}$. Only the photons with frequencies ω_{wave} existed in a waveguide, and these

photons start interacting with the plates inserted inside a waveguide. The optical contribution to the Casimir energy arising from the interaction of these ω_{wave} photons with inserted plates is equal to

$$\begin{aligned}
& -2\pi \sum_{\omega_{wave}} (G_{optical}^D(\mathbf{x}, \mathbf{x}, a, i\omega_{wave}) + G_{optical}^N(\mathbf{x}, \mathbf{x}, a, i\omega_{wave})) = \frac{\pi}{48a} + \\
& + \frac{1}{4} \sum_{n_2, n_3 = -\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \ln \left(1 - \exp \left[-2a \sqrt{\left(\frac{\pi n_2}{b} \right)^2 + \left(\frac{\pi n_3}{c} \right)^2 + p^2} \right] \right) \quad (108)
\end{aligned}$$

(this equality is obtained in analogy to (106) and (107)). After comparison with (91) it is straightforward to rewrite the Casimir energy of the rectangular cavity in terms of optical contributions :

$$\begin{aligned}
E_{cavity}(a, b, c) &= \\
&= \frac{\pi}{48b} + \frac{\pi}{48c} - 2\pi \sum_{\omega_i} (G_{optical}^D(\mathbf{x}, \mathbf{x}, c, i\omega_i) + G_{optical}^N(\mathbf{x}, \mathbf{x}, c, i\omega_i)) - \\
&- 2\pi \sum_{\omega_{plates}} (G_{optical}^D(\mathbf{x}, \mathbf{x}, b, i\omega_{plates}) + G_{optical}^N(\mathbf{x}, \mathbf{x}, b, i\omega_{plates})) - \\
&- 2\pi \sum_{\omega_{wave}} (G_{optical}^D(\mathbf{x}, \mathbf{x}, a, i\omega_{wave}) + G_{optical}^N(\mathbf{x}, \mathbf{x}, a, i\omega_{wave})). \quad (109)
\end{aligned}$$

Here we sum over all eigenfrequencies of the electromagnetic field in 3 cases: when there is an infinite space ($\omega_i = \sqrt{k_x^2 + k_y^2 + k_z^2}$), two parallel plates ($\omega_{plates} = \sqrt{(\pi n_2/b)^2 + k_x^2 + k_z^2}$) and an infinite waveguide ($\omega_{wave} = \sqrt{(\pi n_2/b)^2 + (\pi n_3/c)^2 + (k_x)^2}$).

The first two terms in (109) may have the following geometric interpretation: from (90) it follows that

$$\frac{\pi}{48b} = \pi \sum_{\omega_{mw}} (G_{optical}^D(\mathbf{x}, \mathbf{x}, b, i\omega_{mw}) + G_{optical}^N(\mathbf{x}, \mathbf{x}, b, i\omega_{mw})) \quad (110)$$

$$\frac{\pi}{48c} = \pi \sum_{\omega_{mw}} (G_{optical}^D(\mathbf{x}, \mathbf{x}, c, i\omega_{mw}) + G_{optical}^N(\mathbf{x}, \mathbf{x}, c, i\omega_{mw})), \quad (111)$$

where ω_{mw} is an eigenfrequency of a main wave in a waveguide. So it is possible to express Casimir energies of perfectly conducting rectangular cavities in terms of optical Green's functions only.

It is interesting that the Casimir energy of a perfectly conducting cavity can be written in terms of eigenfrequencies of the electromagnetic field in a free space, between two perfectly conducting plates and inside a perfectly conducting waveguide.

3.7 The experiment

For the experimental check of the Casimir energy for the rectangular cavity one should measure the force somehow. We think about the following possibility: one should insert two parallel perfectly conducting plates inside an infinite perfectly conducting waveguide and measure the force acting on one of the plates as it is being moved through the waveguide. The distance between the inserted plates is a .

To calculate the force on each plate the following gedanken experiment is useful. Imagine that 4 parallel plates are inserted inside an infinite waveguide and then 2 exterior plates are moved to spatial infinities. This situation is exactly equivalent to 3 perfectly conducting cavities touching each other. From the energy of this system one has to subtract the Casimir energy of an infinite waveguide, only then do we obtain the energy of interaction between the interior parallel plates, the one that can be measured in the proposed experiment (the subtraction of the term (72) is just the same subtraction for two parallel plates). Doing so we obtain the attractive force on each interior plate inside the waveguide:

$$F_{attr}(a, b, c) = -\frac{\partial E_{attr}(a, b, c)}{\partial a}, \quad (112)$$

where

$$E_{attr}(a, b, c) = \frac{\pi}{48a} + \frac{1}{4} \sum_{n_2, n_3=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \ln \left(1 - \exp \left[-2a \sqrt{\left(\frac{\pi n_2}{b} \right)^2 + \left(\frac{\pi n_3}{c} \right)^2 + p^2} \right] \right) \quad (113)$$

coincides with (108). We note that our formula (112) for the special case $b = c$ coincides with the formula (6) in reference [35], there it was obtained using a different method and presented in a different mathematical form.

To obtain the energy of interaction between the opposite sides of a *single* cavity one should subtract from the expression (91) the Casimir energy of the same box without these two sides, i.e. one has to subtract from (91) the expression for the Casimir energy of a waveguide of a finite length. To our knowledge the expression for the Casimir energy of a finite length waveguide is not known up to now.

It was often argued that the constant repulsive force (for a fixed cross section) derived from (91) can be measured in experiment. However, without the subtraction just mentioned it is not possible to measure the forces in any realistic experiment, this is why it is not possible to use the expression (91)

directly to calculate the force in the experiments. However, it can be used to derive a measurable in experiments expression for the force between the parallel plates inserted inside an infinite waveguide of the same cross section as the plates.

Using the same technique as before it is possible to generalize our formulas (112), (113) for the case of an infinite waveguide with an arbitrary cross section. The force between the two plates inside this waveguide can be immediately written :

$$F(a) = -\frac{\partial E_{arb}(a)}{\partial a}, \quad (114)$$

$$E_{arb}(a) = \sum_{\omega_{wave}} \frac{1}{2} \ln(1 - \exp(-2a \omega_{wave})), \quad (115)$$

the sum here is over all TE and TM eigenfrequencies ω_{wave} for the waveguide with an arbitrary cross section and an infinite length. Thus it can be said that *the exchange of photons with the eigenfrequencies of a waveguide between the inserted plates always yields the attractive force between the plates.*

To get the free energy $F_{arb}(a, \beta)$ for bosons at nonzero temperatures $\beta = 1/T$ one has to make the substitutions (see Sec. 2.3, the formula (27)):

$$p \rightarrow p_m = \frac{2\pi m}{\beta}, \quad (116)$$

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi} \rightarrow \frac{1}{\beta} \sum_{m=-\infty}^{+\infty}. \quad (117)$$

Thus the free energy describing the interaction of the two parallel perfectly conducting plates inside an infinite perfectly conducting waveguide of an arbitrary cross section has the form:

$$\begin{aligned} F_{arb}(a, \beta) = & \frac{1}{\beta} \sum_{\lambda_{kD}} \sum_{m=-\infty}^{+\infty} \frac{1}{2} \ln \left(1 - \exp(-2a \sqrt{\lambda_{kD}^2 + p_m^2}) \right) + \\ & + \frac{1}{\beta} \sum_{\lambda_{iNeum}} \sum_{m=-\infty}^{+\infty} \frac{1}{2} \ln \left(1 - \exp(-2a \sqrt{\lambda_{iNeum}^2 + p_m^2}) \right), \end{aligned} \quad (118)$$

where λ_{kD}^2 and λ_{iNeum}^2 are eigenvalues of the two-dimensional Dirichlet and Neumann problems (a boundary here coincides with the boundary of each plate inside the waveguide):

$$\Delta^{(2)} f_k(x, y) = -\lambda_{kD}^2 f_k(x, y) \quad (119)$$

$$f_k(x, y)|_{\partial M} = 0, \quad (120)$$

$$\Delta^{(2)}g_i(x, y) = -\lambda_{iNeum}^2 g_i(x, y) \quad (121)$$

$$\left. \frac{\partial g_i(x, y)}{\partial n} \right|_{\partial M} = 0. \quad (122)$$

The attractive force between the plates inside an infinite waveguide of the same cross section at nonzero temperatures is given by:

$$F(a, \beta) = -\frac{\partial F_{arb}(a, \beta)}{\partial a} = -\frac{1}{\beta} \sum_{\omega_{TD}} \frac{\omega_{TD}}{\exp(2a\omega_{TD}) - 1} - \frac{1}{\beta} \sum_{\omega_{TN}} \frac{\omega_{TN}}{\exp(2a\omega_{TN}) - 1}. \quad (123)$$

Here $\omega_{TD} = \sqrt{p_m^2 + \lambda_{kD}^2}$ and $\omega_{TN} = \sqrt{p_m^2 + \lambda_{iNeum}^2}$.

The proof of these results will be presented elsewhere.

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